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Domain defects in systems of two real scalar fields*

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Abstract: In this work we investigate the role of the symmetry of the Lagrangian on the existence of defects in systems of coupled scalar fields. We focus attention mainly on solutions where defects may nest defects. When space is non-compact we find topological BPS and non-BPS solutions that present internal structure. When space is compact the solutions are nontopological sphalerons, which may be nested inside the topological defects. We address the question of classical stability of these topological and nontopological solutions and investigate how the thermal corrections may modify the classical scenario.

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Contents

1. Introduction

In recent years much investigation on topological defects have been done with applications on diverse fields as for instance cosmology and particle physics [1, 2]. In the present paper we are interested in systems of coupled scalar fields where solutions like defects inside defects may appear. Such situation was first investigated in the work of Witten [3], within the context of superconducting strings — see also the work of Lazarides and Shafi [4] and of MacKenzie [5]. Several aspects of the possibility of topological defects being nested inside topological defects are considered in refs. [6]–[10], and in refs. [11]–[15] other related issues are also investigated.

To introduce defects inside defects one needs to consider systems of at least two fields, one of the fields being able to describe a specific defect, which will respond for nesting the other defect in its interior. The system of two fields should then be able to accommodate some defect and, in the core of this defect, be reduced to a system of the second field that is still able to generate the other defect. Since these defects are usually topological, and since topological defects appear in systems presenting spontaneous symmetry breaking, the complete system usually presents $G_1 \times G_2$ symmetry, for instance $\mathrm{U}(1) \times \mathrm{U}(1)$ in the case of the superconducting strings examined by Witten [3] or $Z_2 \times Z_2$ domain walls, as considered for instance in the recent works [7, 8, 9, 10].

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Although systems of two coupled fields engendering $G_1 \times G_2$ symmetry are known to support defects inside defects, we may wonder if defects with internal structures appear after changing the symmetry of the system. Stated differently, we may ask how the picture of defects inside defects modifies when one changes the symmetry of the underlying lagrangean, enlarging or reducing the $G_1 \times G_2$ group. This issue is of general interest and may find direct applications in cosmology and in condensed matter. In the present work we address this and other related questions in systems of two real scalar fields, described via the lagrangean density

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi + \frac{1}{2}\partial_\alpha\chi\partial^\alpha\chi - U(\phi,\chi)\,. \tag{1.1}$$

Although we are working in $3+1$ space-time dimensions we can search for static solutions in the form $\phi = \phi(x)$ and $\chi = \chi(x)$, depending on only one ($x^1 = x$) of the three spatial coordinates. In this case the equations of motion are given by

$$\frac{d^2\phi}{dx^2} = \frac{\partial U}{\partial\phi}\,, \tag{1.2}$$

$$\frac{d^2\chi}{dx^2} = \frac{\partial U}{\partial\chi}\,. \tag{1.3}$$

We are interested in a special class of systems, in which the potential is determined by

$$U(\phi,\chi) = \frac{1}{2}\left(\frac{\partial H}{\partial\phi}\right)^2 + \frac{1}{2}\left(\frac{\partial H}{\partial\chi}\right)^2. \tag{1.4}$$

It is defined in terms of a smooth but otherwise arbitrary function $H = H(\phi,\chi)$ and now the equations of motion become

$$\frac{d^2\phi}{dx^2} = H_\phi H_{\phi\phi} + H_\chi H_{\chi\phi}\,, \tag{1.5}$$

$$\frac{d^2\chi}{dx^2} = H_\phi H_{\phi\chi} + H_\chi H_{\chi\chi}\,, \tag{1.6}$$

where H_ϕ stands for $\partial H/\partial\phi$, and so forth. The advantage of considering such potential is that the energy of static configurations is minimized to the value

$$E_B = \Big|H[\phi(+\infty),\chi(+\infty)] - H[\phi(-\infty),\chi(-\infty)]\Big|\,, \tag{1.7}$$

which depends just on the asymptotic values of the field configurations. Furthermore, the second-order differential equations of motion are now solved by field configurations that obey the first-order equations

$$\frac{d\phi}{dx} = H_\phi\,, \tag{1.8}$$

$$\frac{d\chi}{dx} = H_\chi\,. \tag{1.9}$$

This result can be shown [11] to follow the idea of Bogomol'nyi, Prasad and Sommerfield (BPS) [16] in the case of coupled real scalar fields. This means that the potential in Eq. (1.4) can be considered as describing the bosonic sector of a larger supersymmetric theory, so that supersymmetric extensions of the lagrangean (1.1) can be readly constructed, as was considered for instance in ref. [10, 12] as well as in several other investigations [13, 14, 15].

The present work is organized as follows. In the next sec. 2 we search for defects inside defects starting with a general potential. There we show that defects may engender internal structure only when the system presents parity symmetry of the type $Z_2 \times Z_2$. Once this is stablished we write down explicit solutions, examining both the second-order equations of motion and the first-order equations that give BPS solutions. We also consider the possibility of dealing with compact space, to obtain periodic solutions similar to the sphaleron solutions discussed by Manton and Samols [17]. Such solutions are known to be useful for discussing violation of the barionic number, and may contribute to applications in cosmology. In sec. 3 we investigate classical or linear stability of the solutions introduced in sec. 2. Although some of these solutions were already discussed in refs. [8, 9], a complete investigation of their stability is still missing and will be done in this work. In sec. 4 we introduce finite temperature contributions to investigate how the thermal effects may change the classical scenario. We end this paper in sec. 5, where we present comments and conclusions.

2. Symmetry and defects inside defects

We start with the simplest case in which a theory presents a Z_2 parity symmetry with nontrivial vacuum states for a single field ϕ. For

$$U(\phi) = \frac{1}{2} H_\phi^2 \,, \tag{2.1}$$

the standard ϕ^4 model is obtained with the function

$$H(\phi) = \lambda \left(\frac{1}{3}\phi^3 - a^2\phi \right) . \tag{2.2}$$

A natural generalization for two coupled fields, in a renormalizable theory in (3+1) dimensions can be written as

$$H(\phi,\chi) = \lambda \left(\frac{1}{3}\phi^3 - a^2\phi \right) + \mu\phi\chi^2 + \nu\phi^2\chi + \sigma \left(\frac{1}{3}\chi^3 - b^2\chi \right) , \tag{2.3}$$

where λ, μ, ν, σ, a and b are real parameters. Here we have the general potential

$$\begin{aligned} U(\phi,\chi) = {} & \frac{1}{2}(\lambda^2+\nu^2)\phi^4 + 2\nu(\lambda+\mu)\phi^3\chi + (\lambda\mu + 2\nu^2 + \sigma\nu + 2\mu^2)\phi^2\chi^2 + \\ & + 2\mu(\nu+\sigma)\phi\chi^3 + \frac{1}{2}(\mu^2+\sigma^2)\chi^4 - (\lambda^2 a^2 + \sigma b^2 \nu)\phi^2 - (\lambda a^2 \mu + \sigma^2 b^2)\chi^2 - \\ & - 2(\lambda a^2 \nu + \sigma b^2 \mu)\phi\chi + \frac{1}{2}(\lambda^2 a^4 + \sigma^2 b^4) \,. \end{aligned} \tag{2.4}$$

We now impose that the theory presents $Z_2 \times Z_2$ symmetry. To ensure invariance of the lagrangean under the independent transformations $\phi \to -\phi$ and $\chi \to -\chi$ we set

$$\nu(\lambda + \mu) = 0\,, \tag{2.5}$$

$$\mu(\nu + \sigma) = 0\,, \tag{2.6}$$

$$\lambda\nu a^2 + \sigma\mu b^2 = 0\,. \tag{2.7}$$

There are four possibilities of satisfying the above conditions; they are:

$$\begin{aligned}
&\nu = \sigma = 0\,,\\
&\lambda = \mu = 0\,,\\
&\nu = \mu = 0\,,\\
&\lambda = -\mu\,; \qquad \nu = -\sigma\,; \qquad a^2 = -b^2\,.
\end{aligned} \tag{2.8}$$

The first possibility gives a model for which the function $H(\phi, \chi)$ reads

$$H_1(\phi, \chi) = \lambda\left(\frac{1}{3}\phi^3 - a^2\phi\right) + \mu\phi\chi^2\,, \tag{2.9}$$

which was discussed recently in the literature [7, 8]. This is a tipical example, shown to present topological solutions of the type of defects inside defects. The second possibility leads to a model that is described by

$$H_2(\phi, \chi) = \sigma\left(\frac{1}{3}\chi^3 - b^2\chi\right) + \nu\phi^2\chi\,, \tag{2.10}$$

which is equivalent to the previous one, as can be seen by simply exchanging the fields ϕ and χ and redefining the parameters σ, ν and b. The last two possibilities give uninteresting models, at least from the point of view of defects inside defects: the third possibility leads to a system of two decoupled scalar fields and the fourth one severely restricts the number of nontrivial vacua, forbidding the presence of defects inside defects.

2.1 Topological solutions

We consider the equations of motion (1.5) and (1.6) for the general model described by $H(\phi, \chi)$, given by eq. (2.3). We have

$$\begin{aligned}
\frac{d^2\phi}{dx^2} &= \left[\lambda(\phi^2 - a^2) + \mu\chi^2 + 2\nu\phi\chi\right](2\lambda\phi + 2\nu\chi) +\\
&\quad + \left[2\mu\phi\chi + \nu\phi^2 + \sigma(\chi^2 - b^2)\right](2\mu\chi + 2\nu\phi)\,,
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\frac{d^2\chi}{dx^2} &= \left[\lambda(\phi^2 - a^2) + \mu\chi^2 + 2\nu\phi\chi\right](2\mu\chi + 2\nu\phi) +\\
&\quad + \left[2\mu\phi\chi + \nu\phi^2 + \sigma(\chi^2 - b^2)\right](2\mu\phi + 2\sigma\chi)\,.
\end{aligned} \tag{2.12}$$

To find defects inside defects we have to search for pairs of solutions like (ϕ,0) and (0,χ). For the first pair we set $\chi = 0$ and then the above equations reduce to

$$\frac{d^2\phi}{dx^2} = \left[2\lambda^2(\phi^2 - a^2) + 2\nu^2\phi^2 - 2\sigma\nu b^2\right]\phi\,, \tag{2.13}$$

and

$$\nu(\lambda + \mu)\phi^2 - (\lambda\nu a^2 + \mu\sigma b^2) = 0\,. \tag{2.14}$$

For the second pair we have $\phi = 0$ and then eqs. (2.11) and (2.12) read

$$\frac{d^2\chi}{dx^2} = \left[2\sigma^2(\chi^2 - b^2) - 2\mu\lambda a^2 + 2\mu^2\chi^2\right]\chi\,, \tag{2.15}$$

and

$$\mu(\nu + \sigma)\chi^2 - (\lambda\nu a^2 + \mu\sigma b^2) = 0\,. \tag{2.16}$$

Eqs. (2.14) and (2.16) show that solutions like defects inside defects are only possible for

$$\nu(\lambda + \mu) = 0, \qquad \mu(\nu + \sigma) = 0\,, \qquad \lambda\nu a^2 + \mu\sigma b^2 = 0\,, \tag{2.17}$$

Interestingly, these conditions are exactly the conditions (2.5)–(2.7), needed to ensure the $Z_2 \times Z_2$ parity symmetry of the system. This means that only the generating functions $H_1(\phi,\chi)$ and $H_2(\phi,\chi)$ may admit solutions describing defects inside defects.

The explicit solutions can be found by solving the remaining equations. For the system described by H_1 we take $\nu = \sigma = 0$ and rewrite eqs. (2.13) and (2.15) as: $\chi = 0$ and

$$\frac{d^2\phi}{dx^2} = 2\lambda^2(\phi^2 - a^2)\phi\,, \tag{2.18}$$

or $\phi = 0$ and

$$\frac{d^2\chi}{dx^2} = 2\mu^2(\chi^2 - ra^2)\chi\,, \tag{2.19}$$

where we have set $\lambda/\mu = r$. There are pairs of solutions

$$\phi(x) = a\tanh(\lambda a x)\,; \qquad \chi = 0\,, \tag{2.20}$$

and

$$\chi(x) = a\sqrt{r}\tanh(\mu\sqrt{r}ax)\,; \qquad \phi = 0\,. \tag{2.21}$$

These solutions present the feature of vanishing at its own core, that is, for $x \to 0$ we get $\phi(x) \to 0$ and $\chi(x) \to 0$. This is important for introducing defects inside defects in $3+1$ dimensions, since in this case we can choose $\phi = \phi(x)$ and $\chi = \chi(y)$, say, to make topological defect appear in the core of topological defect. The issue of choosing the field to host the second field depends on the parameters of the system, and follows as in the former investigation [8] — see also ref. [10] for similar considerations in 2+1 dimensions.

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Similarly, we can also find solutions like defects inside defects in the other system, defined via $H_2(\phi,\chi)$. In this case the pairs of solutions are given by

$$\phi(x) = b\sqrt{\frac{\sigma}{\nu}}\tanh\left(\nu\sqrt{\frac{\sigma}{\nu}}bx\right)\,; \qquad \chi = 0\,, \tag{2.22}$$

and

$$\chi(x) = b\tanh(\sigma bx)\,; \qquad \phi = 0\,. \tag{2.23}$$

Also they may be used to build defects inside defects in the 3+1 dimensional system.

2.2 BPS solutions

Let us now consider the first-order equations (1.8) and (1.9) for the general H given by eq. (2.3). Here we have

$$\frac{d\phi}{dx} = \lambda(\phi^2 - a^2) + \mu\chi^2 + 2\nu\phi\chi\,, \tag{2.24}$$
$$\frac{d\chi}{dx} = \sigma(\chi^2 - b^2) + \nu\phi^2 + 2\mu\phi\chi\,. \tag{2.25}$$

We set $\chi = 0$ to get to

$$\frac{d\phi}{dx} = \lambda(\phi^2 - a^2)\,, \tag{2.26}$$

and

$$\nu\phi^2 - \sigma b^2 = 0\,. \tag{2.27}$$

We choose $\nu = \sigma = 0$ to get to the pair of solutions

$$\phi(x) = a\tanh\lambda ax\,; \qquad \chi = 0\,. \tag{2.28}$$

Analogously, in the case $\phi = 0$ we have

$$\frac{d\chi}{dx} = \sigma(\chi^2 - b^2)\,, \tag{2.29}$$

and

$$\mu\chi^2 - \lambda a^2 = 0\,. \tag{2.30}$$

We impose $\lambda = \mu = 0$ to obtain

$$\chi(x) = b\tanh\sigma bx\,; \qquad \phi = 0\,. \tag{2.31}$$

These are BPS defects, since they arise from the first-order equations (1.8) and (1.9).

It is interesting to realize that the above BPS solutions cannot be solutions of a single system, because the conditions $\nu = \sigma = 0$ and $\lambda = \mu = 0$ cannot be implemented simultaneously in the general H given by eq. (2.3). This conclusion is in agreement with results of dynamical systems, since one knows from unicity of solutions that any two orbits never cross each other in configuration space. And

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we recall that the pair of first-order equations (2.24) and (2.25) can be seen as a dynamical system. However, we can have the BPS defect (2.28) in the system defined via $H_1(\phi,\chi)$, and the BPS defect (2.31) in the other system, defined via $H_2(\phi,\chi)$.

The above solutions have been investigated before [8, 9], but here we have a stronger result, showing that systems engendering solutions like defects inside defects must present $Z_2\times Z_2$ symmetry. Furthermore, only one of the solutions can be of the BPS type. Since the BPS solution is stable [11], we still have to investigate stability of the other solution to talk about stability of defects with internal structure.

For completeness let us investigate the presence of other BPS solutions in the system defined by $H_1(\phi,\chi)$, for instance. In this case the first-order equations are

$$\frac{d\phi}{dx} = \lambda(\phi^2-a^2)+\mu\chi^2\,, \tag{2.32}$$

$$\frac{d\chi}{dx} = 2\mu\phi\chi\,. \tag{2.33}$$

An interesting pair of BPS solutions was already obtained in [11]. It is

$$\phi(x) = -a\tanh(2\mu a x)\,, \tag{2.34}$$

$$\chi(x) = \pm a\sqrt{r-2}\,\mathrm{sech}(2\mu a x)\,. \tag{2.35}$$

It is valid for $r=\lambda/\mu>2$ and obeys

$$\phi^2+\frac{1}{r-2}\,\chi^2 = a^2\,, \tag{2.36}$$

which defines a semi-ellipsis in configuration space. We recall that at $r=3$ the amplitude of the two above solutions degenerate to a single value, the orbit changes to a semi-circle and the corresponding stability can be implemented analytically [23].

This system presents peculiar behavior at two other values of r. For $r=1$ the symmetry changes from $Z_2\times Z_2$ to Z_4, but now the system degenerates into two uncoupled systems of one field [8]. For $r=-1$ the vacuum states at the χ axis desapear, and although there is still the pair of BPS solutions given by eq. (2.28), it is possible to show that solutions like the above one, describing a finite orbit connecting the two vacuum states $(-a,0)$ and $(a,0)$ with non-vanishing χ cannot be present anymore. The proof follows after recognizing that for $r=-1$ the first-order equations (2.32) and (2.33) implies that

$$\chi\,\left(\phi^2-\frac{1}{3}\chi^2-a^2\right) = 0\,. \tag{2.37}$$

This restriction shows that $\chi=0$ or

$$\phi^2 = a^2+\frac{1}{3}\,\chi^2\,. \tag{2.38}$$

The first condition $\chi=0$ gives exactly the case of the straight line connecting the vacua $(-a,0)$ and $(a,0)$. The other condition eq. (2.38) shows that for $\chi\neq 0$ no finite orbit can connect the vacuun states anymore.

2.3 Periodic solutions

Let us investigate the same systems given by H_1 and H_2, but now searching for periodic solutions of the second-order equations (1.2) and (1.3) that appear when the line becomes a circle, a compact space. In this case the equations of motion do not change but the boundary conditions are now periodic ones, to respect the topology of the circle. The solutions here are like the sphalerons solutions introduced in [17] in the case of a single field — see also [18, 19] and references therein for further details. For generality we shall assume that the fields ϕ and χ are periodic according to

$$\begin{aligned}\phi(x) &= \phi(x+L_1)\,,\\ \chi(x) &= \chi(x+L_2)\,.\end{aligned} \tag{2.39}$$

Evidently, when these fields are immersed in a one-dimensional space we should impose that $L_1 = L_2$, but in higher dimensions we can search for solutions with $\phi = \phi(x)$ and $\chi = \chi(y)$, for instance, and so we can allow $L_1 \neq L_2$ and this is the case we shall consider below. Here the solutions will be given in terms of the Jacobi elliptic function $\mathrm{sn}(x)$, which has period $4K(k_i)$ determined by the elliptic quarter period $K(k_i)$ which obeys ($k_i \in (0,1)$, $i = 1,2$) [20]

$$\frac{1}{2}\pi = K(0) \leq K(k_i) \leq K(1) = \infty\,. \tag{2.40}$$

For the model H_1 we get the solutions: $\chi = 0$ and

$$\phi(x) = a k_1\, b(k_1)\, \mathrm{sn}\left(a\lambda b(k_1)\, x\right), \tag{2.41}$$

and also $\phi = 0$ and

$$\chi(x) = a k_2 \sqrt{r}\, b(k_2)\, \mathrm{sn}\left(a\mu\sqrt{r}\, b(k_2)\; x\right), \tag{2.42}$$

where

$$b(k_i) = \sqrt{\frac{2}{1+k_i^2}}\,, \qquad i = 1,2\,. \tag{2.43}$$

The stability of these solutions will be studied in detail in the next section. Analogous solutions for H_2 can also be found by just changing $\phi \to \chi$ and redefining the parameters conveniently.

In the limit $k_i \to 1$ the radius of the circle goes to infinity, $\mathrm{sn}(b(k_i)x) \to \tanh(x)$, and the above solutions recover the corresponding basic kink solutions we have already found in the previous subsections. Here we notice that in two or more space dimensions we may set $k_1 \to 1$ or $k_2 \to 1$, to make periodic or topological defects to host topological or periodic defects, allowing for new pictures of defects inside defects. As one can check, none of the above periodic solutions of the second-order equations of motion solves the first-order equations, despite the fact that the limit $k_i \to 1$ recover all the topological solutions including the BPS ones.

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3. Linear stability

In this section we investigate the stability of the solutions presented in the former sec. 2. Evidently, a general study of the stability of static solutions of nonlinear equations is a highly nontrivial matter, but the simplest situation known as classical or linear stability may be performed and will be analysed with some detail below. Firstly we recall that the solutions (2.20) and (2.23) also solve the first-order equations, and so are linearly stable [11]. All other solutions should be investigated with respect to their stability.

We start by writing the time-dependent fields in the usual form [21, 22]

$$\begin{aligned} \phi(x,t) &= \phi(x) + \eta(x,t)\,, \\ \chi(x,t) &= \chi(x) + \zeta(x,t)\,, \end{aligned} \tag{3.1}$$

where $\eta(x,t)$ and $\zeta(x,t)$ are the time-dependent perturbations, small when compared to $\phi(x)$ and $\chi(x)$, the static solutions of the second-order equations of motion. Since we are considering fluctuations about static solutions, we can always write

$$\begin{aligned} \eta(x,t) &= \sum_n \eta_n(x)\cos(w_n t)\,, \\ \zeta(x,t) &= \sum_n \zeta_n(x)\cos(w_n t)\,. \end{aligned} \tag{3.2}$$

Substituting these expressions into the time-dependent Euler-Lagrange equations of motion we get the following Schrödinger equation

$$\left(-\mathbf{1}\frac{d^2}{dx^2} + \mathbf{M}\right)\begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix} = w_n^2 \begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix}, \tag{3.3}$$

where $\mathbf{1}$ is the 2×2 identity matrix and

$$\mathbf{M} = \begin{pmatrix} U_{\phi\phi} & U_{\phi\chi} \\ U_{\chi\phi} & U_{\chi\chi}\end{pmatrix}\,. \tag{3.4}$$

The derivatives of the potential $U(\phi,\chi)$ should be calculated at the classical values $\phi(x)$ and $\chi(x)$, and we recall that stability means absence of negative w_n^2.

The standard analysis of stability usually considers diagonalizing the above matrix $\mathbf{M}$, but this may make the problem untractable analytically. We illustrate this situation by following the lines of ref. [23], where instead of considering the matrix involving derivatives of the potential $U(\phi,\chi)$, one considers another one, involving derivatives of $H(\phi,\chi)$. This procedure follows naturally from the fact that the first order equations can be used to simplify investigations concerning the second-order equations, at least when the classical configurations also solve the first-order differential equations. For BPS solutions this is implemented after introducing the first-order operators

$$S_{\pm} = \pm\mathbf{1}\frac{d}{dx} + \mathbf{m}\,, \tag{3.5}$$

where $\mathbf{m}$ is the matrix

$$\mathbf{m} = \begin{pmatrix} H_{\phi\phi} & H_{\phi\chi} \\ H_{\chi\phi} & H_{\chi\chi} \end{pmatrix}, \tag{3.6}$$

so that the stability equation (3.3) can be rewritten as

$$\left[-\mathbf{1}\frac{d^2}{dx^2} + \frac{d\mathbf{m}}{dx} + \mathbf{m}^2\right]\begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix} = S_+S_-\begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix} \tag{3.7}$$

$$= w_n^2\begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix}. \tag{3.8}$$

In (3.7) the Schrödinger-like operator presents the form S_+S_-. However, in eq. (3.5) the first-order operators are such that $S_\pm^\dagger = S_\mp$. Thus we can write

$$w_n^2 = \langle n|S_+S_-|n\rangle = \langle n|S_-^\dagger S_-|n\rangle = ||S_-|n\rangle||^2 \geq 0\,, \tag{3.9}$$

where $|n\rangle$ stands for the (orthonormalized) state

$$|n\rangle = \begin{pmatrix}\eta_n \\ \zeta_n\end{pmatrix}. \tag{3.10}$$

This shows explicitly that the Schrödinger-like operator in eq. (3.7) is positive semi-definite, and so the corresponding eigenvalues w_n^2 are non-negative — see ref. [11] for further details. The BPS solutions are stable, but if we want to find the eigenvalues explicitly we need to diagonalize the corresponding potential. This diagonalization for BPS solutions can be given in terms of the diagonalization of the simpler matrix $\mathbf{m}$. In this case the elements of the diagonalized matrix are

$$\lambda_\pm = \frac{1}{2}(H_{\phi\phi} + H_{\chi\chi}) \pm \frac{1}{2}\sqrt{(H_{\phi\phi} - H_{\chi\chi})^2 + 4{H_{\phi\chi}}^2}\,, \tag{3.11}$$

and the corresponding Schrödinger-like equations are given by

$$\left(-\frac{d^2}{dx^2} + \frac{d\lambda_\pm}{dx} + (\lambda_\pm)^2\right)\eta_n^\pm = w_n^2\,\eta_n^\pm\,. \tag{3.12}$$

The problem of solving the above equations analytically is now related to the presence of the square root in the potential of the corresponding Schrödinger equations. To mantain the investigation analytical, we should get rid of the square root in eq. (3.11), and this can be done by imposing one of the following conditions

$$H_{\phi\chi} = 0\,, \tag{3.13}$$

$$H_{\phi\phi} = H_{\chi\chi}\,, \tag{3.14}$$

$$H_{\phi\phi}H_{\chi\chi} = {H_{\phi\chi}}^2\,. \tag{3.15}$$

For the general model, the above condition (3.13) can be satisfied with the following possibilities

$$\mu = \nu = 0\,, \tag{3.16}$$

$$\nu = 0 \text{ and } \chi = 0\,, \tag{3.17}$$

$$\mu = 0 \text{ and } \phi = 0\,. \tag{3.18}$$

The possibility (3.16) must be discarded since it decouples the two fields. The other two possibilities (3.17) and (3.18) can be considered and here ϕ in case (3.17) and χ in case (3.18) may describe topological domain defects of the BPS type, but in these cases no defect is allowed to appear inside it.

The second possibility of eliminating the square root from the stability equation is $H_{\phi\phi} = H_{\chi\chi}$. Since the generalized model presents $H_{\phi\phi} = 2\lambda\phi + 2\nu\chi$ and $H_{\chi\chi} = 2\mu\phi + 2\sigma\chi$ we see that the condition (3.14) is obtained by imposing $\lambda = \mu$ and $\sigma = \nu$. In this case we have

$$H_s = \mu\left(\frac{1}{3}\phi^3 - a^2\phi\right) + \mu\phi\chi^2 + \nu\phi^2\chi + \nu\left(\frac{1}{3}\chi^3 - b^2\chi\right). \tag{3.19}$$

This function can be written in the form $H_s = \bar{H}_1 + \bar{H}_2$, where

$$\begin{aligned} \bar{H}_1 &= \mu\left(\frac{1}{3}\phi^3 - a^2\phi\right) + \mu\phi\chi^2\,, \\ \bar{H}_2 &= \nu\left(\frac{1}{3}\chi^3 - b^2\chi\right) + \nu\chi\phi^2\,. \end{aligned} \tag{3.20}$$

In this case, however, we follow [8] to see that each $\bar{H}_i\,(i = 1, 2)$ can be written in terms of $\phi_\pm = 2^{-1/2}\,(\phi \pm \chi)$, in a form that decouple the two fields, and so also does the full system described by H_s. Thus we see that the general model does not give any system of two coupled fields under the additional condition $H_{\phi\phi} = H_{\chi\chi}$.

The third and last condition (3.15) implies

$$\lambda\mu\phi^2 + \sigma\nu\chi^2 + (\lambda\sigma + \mu\nu)\phi\chi = \nu^2\phi^2 + \mu^2\chi^2 + 2\mu\nu\phi\chi\,, \tag{3.21}$$

which is satisfied when we identify $\lambda\mu = \nu^2$ and $\sigma\nu = \mu^2$, which also implies $\lambda\sigma = \mu\nu$. These conditions lead to a simpler system, but such system describes no system of two coupled fields anymore.

We see that none of the conditions (3.13)–(3.15) leads to systems of coupled fields. However, eq. (3.11) shows another possibility, which appears as follows: we investigate the quantity

$$(H_{\phi\phi} - H_{\chi\chi})^2 + 4\,H_{\phi\chi} \tag{3.22}$$

with the aim of eliminating the square root in eq. (3.11), restricting the set of parameters that defines $H(\phi,\chi)$. This possibility is particular, and can only be implemented in specific systems, where $H(\phi,\chi)$ is explicitly known. For instance, in ref. [23] one uses this reasoning to present analytical calculations that allow obtaining the discrete eigenvalues corresponding to BPS solutions of systems like the ones described by (2.9) or (2.10), in a specific region of the space of parameters.

The above results show that although the BPS solutions are stable, in general it is not ease to know the corresponding eigenvalues explicitly. Evidently, we can always give up the analytical procedure to solve Schrödinger-like equations like eq. (3.12) numerically, but this is out of the scope of the present work. From the point of view of defects inside defects, the important information is that the BPS solutions are stable configurations, and so they can be considered stable defects to host non-BPS defects. The stability of the non-BPS defects will be investigated in the next subsection.

The result that the general system does not give any system of two coupled fields for $H_{\phi\phi} = H_{\chi\chi}$ can be generalized in the following way. We introduce the new fields $\phi_\pm = 2^{-1/2}\,(\phi \pm \chi)$ and use the equations of motion (1.5) and (1.6) to obtain

$$\begin{aligned} \frac{d^2\phi_+}{dx^2} &= H_+ H_{++} + H_- H_{-+}\,, \\ \frac{d^2\phi_-}{dx^2} &= H_+ H_{+-} + H_- H_{--}\,. \end{aligned} \tag{3.23}$$

Here we are using the notation $H_+ = \partial H/\partial\phi_+$, $H_- = \partial H/\partial\phi_-$, and so forth. Derivatives of the function H are related by

$$\begin{aligned} H_\phi &= \frac{1}{\sqrt{2}}\,(H_+ + H_-)\,, \\ H_\chi &= \frac{1}{\sqrt{2}}\,(H_+ - H_-)\,, \end{aligned} \tag{3.24}$$

and also

$$H_{\phi\phi} = \frac{1}{2}\,H_{++} + \frac{1}{2}\,H_{--} + \frac{1}{2}\,H_{+-} + \frac{1}{2}\,H_{-+}\,, \tag{3.25}$$

$$H_{\chi\chi} = \frac{1}{2}\,H_{++} + \frac{1}{2}\,H_{--} - \frac{1}{2}\,H_{+-} - \frac{1}{2}\,H_{-+}\,, \tag{3.26}$$

$$H_{\phi\chi} = \frac{1}{2}\,H_{++} - \frac{1}{2}\,H_{--} + \frac{1}{2}\,H_{-+} - \frac{1}{2}\,H_{+-}\,, \tag{3.27}$$

$$H_{\chi\phi} = \frac{1}{2}\,H_{++} - \frac{1}{2}\,H_{--} + \frac{1}{2}\,H_{+-} - \frac{1}{2}\,H_{-+}\,. \tag{3.28}$$

We use (3.27) and (3.28) to see that the condition $H_{\phi\chi} = H_{\chi\phi}$ now becomes $H_{+-} = H_{-+}$. Furthermore, from (3.25) and (3.26) we realize that if one further imposes the condition $H_{\phi\phi} = H_{\chi\chi}$ one gets $H_{+-} = H_{-+} = 0$. This result is general, and implies that $H(\phi_+,\phi_-)$ can be written as the sum of two functions, one depending on ϕ_+ and the other on ϕ_-. The lagrangean density is then reduced to a sum of two lagrangean densities, one for the field ϕ_+ and the other for ϕ_-. The system decouples into two systems of a single field each one, and so it does not describe two coupled fields anymore.

3.1 Stability of the topological solutions

The pair of solutions with $\phi = 0$ and χ given by eq. (2.21) constitutes a pair of non-BPS solutions, that is, it does not obey the corresponding first-order equations. Then its stability should be investigated explicitly, and we do it now since it was not done in the former works [7, 8]. Here, we have that $U_{\phi\chi}$ vanishes at the classical values with $\phi = 0$, and we are left with the following uncoupled Schrödinger-like equations, after appropriately rescaling the space coordinate,

$$\left(\frac{d^2}{dz^2} + \frac{w_n^2}{r\mu^2 a^2} - 4 + 2(2+r)\operatorname{sech}^2 z\right)\eta_n(z) = 0\,,$$
$$\left(\frac{d^2}{dz^2} + \frac{w_m^2}{r\mu^2 a^2} - 4 + 6\operatorname{sech}^2 z\right)\zeta_m(z) = 0\,, \tag{3.29}$$

which are well known modified Posch-Teller equations [24] whose eigenvalues are given by

$$\frac{w_n^2}{r\mu^2 a^2} = 4 - \left[\sqrt{2(2+r)+\frac{1}{4}} - \left(n+\frac{1}{2}\right)\right]^2,$$
$$\frac{w_m^2}{r\mu^2 a^2} = 4 - \left[\frac{5}{2} - \left(m+\frac{1}{2}\right)\right]^2, \tag{3.30}$$

where $n = 0, 1, \ldots, < \sqrt{2(2+r)+1/4} - 1/2$ and $m = 0, 1$. Since $r > 0$ we find the following restriction on the solution (2.21) of the parity preserving model H_1

$$0 < r \equiv \frac{\lambda}{\mu} \leq 1\,. \tag{3.31}$$

However, since we assume that $r \neq 1$, in order to keep the fields coupled, we have $r \in (0, 1)$ as the range of values that ensure stability of the non-BPS solution (2.21). Similar conclusions can be found for the other system, with H_2.

3.2 Stability of the periodic solutions

We now discuss the stability of the periodic solutions (2.41) and (2.42). The general discussion on linear stability already presented applies equally well here, the difference in the present case is that the equations that appear for the fluctuations are now of the Lamé type

$$\left\{\frac{d^2}{dz^2} + h - N(N+1)\mathrm{sn}^2(z)\right\} f(z) = 0\,. \tag{3.32}$$

As we are going to show, some of these equations are solved by the Lamé polynomials, since N can be identified with integer. But there are others for which N is not integer and so the exact solutions will be not completely known [25]. Evidently, for each

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solution we have two equations of stability, the first describing fluctuation in ϕ and the second in χ. For the periodic solution (2.41) the stability equations are

$$\left\{\frac{d^2}{dz^2}+(1+k^2)\left(\frac{\omega^2}{2a^2\mu^2r^2}+1\right)-6k^2\mathrm{sn}^2(z)\right\}\eta(z)=0\,,$$

$$\left\{\frac{d^2}{dz^2}+(1+k^2)\left(\frac{\omega^2}{2a^2\mu^2r^2}+\frac{1}{r}\right)-\frac{2}{r}(1+\frac{2}{r})k^2\mathrm{sn}^2(z)\right\}\zeta(z)=0\,, \tag{3.33}$$

where we omit indices on eigenfunctions and eigenvalues and also on k_1 and k_2, for simplicity. The first stability equation is of the Lamé type with $N=2$, so that there are $2N+1=5$ solutions whose eigenvalues are

$$\frac{w^2}{r^2a^2\mu^2b^2(k)}=3,\ 3k^2,\ 0,\ 1+k^2\pm 2\sqrt{1-k^2+k^4}\,. \tag{3.34}$$

Here, instability appears because of the last eigenvalue with the minus sign, which is always negative for any k since $0\le k^2<1$.

The second stability equation is also of Lamé type but now $N=2/r$ is not an integer, in general. The solutions for this case are not completely known but are expressible as series of elliptic functions which truncate to a polynomial when N becomes an integer. Also, the corresponding eigenvalues for h are not known in general, except for a few cases like $N=1/2$ or $3/2$. So, for special choices of the ratio r

r	4	2	4/3	1	2/3	...
N	1/2	1	3/2	2	3	...

we are able to find explicit solutions and the corresponding eigenvalues. Let us analyse some of these possibilities. For $r=2$, $N=1$ the eigenvalues are:

$$\frac{\omega^2}{4a^2\mu^2b^2(k)}=1,\ \frac{1-k}{1+k},\ \frac{k-1}{1+k}\,, \tag{3.35}$$

which are non-negative except for the last one. For the case $r=1$, $N=2$ the eigenvalues are

$$\frac{\omega^2}{a^2\mu^2b^2(k)}=3,3k^2,0,1+k^2\pm 2\sqrt{1-k^2+k^4}\,, \tag{3.36}$$

which are positive except for the last one with the minus sign, which is negative for any $k\in(0,1)$. Let us also comment on one case where N in half-integer: we choose it to be $N=1/2$, which corresponds to $r=4$; in this case the eigenvalue is zero, exactly.

A similar situation appears for the study of the stability of the second periodic solution (2.42), for which the stability equations are

$$\left\{\frac{d^2}{dz^2}+(1+k^2)\left(\frac{\omega^2}{2a^2\mu^2r}+1\right)-6k^2\mathrm{sn}^2(z)\right\}\zeta(z)=0\,,$$

$$\left\{\frac{d^2}{dz^2}+(1+k^2)\left(\frac{\omega^2}{2a^2\mu^2r}+\frac{1}{r}\right)-2(2+r)k^2\mathrm{sn}^2(z)\right\}\eta(z)=0\,. \tag{3.37}$$

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Here we also find that $N = 2$ for the first stability equation, and in this case the eigenvalues are

$$\frac{\omega^2}{a^2\mu^2 r b^2(k) k^2} = 3,\ 3k^2,\ 0,\ 1 + k^2 \pm 2\sqrt{1 - k^2 + k^4}\,. \tag{3.38}$$

Analogously to the previous case, the second stability equation for the solution (2.42) implies

$$N = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8(2 + r)}\,, \tag{3.39}$$

so that in general we will have no closed solution except for special values of the ratio r

r	1	4	8	13	...
N	2	3	4	5	...

Note that lower values of N are not allowed since we imposed $r > 0$. However, if we choose $r = 1$, $N = 2$ we have the eigenvalues

$$\frac{\omega^2}{a^2\mu^2 b^2(k)} = 3,\ 3k^2,\ 0,\ 1 + k^2 \pm 2\sqrt{1 - k^2 + k^4}\,, \tag{3.40}$$

and the periodic solution (2.42) is also unstable, as expected.

We notice that the above stability equations recover the corresponding stability equations of the former subsection in the limit $k \to 1$. Since $\mathrm{sn}^2 z \to \tanh^2 z = 1 - \mathrm{sech}^2 z$, we see that the operator

$$\frac{d^2}{dz^2} + h_k - k^2 p\, \mathrm{sn}^2 z\,, \tag{3.41}$$

changes to

$$\frac{d^2}{dz^2} + (h_1 - p) + p\, \mathrm{sech}^2 z\,. \tag{3.42}$$

However, there are subtleties in the limit $k \to 1$ for the eigenfunctions and eigenvalues of the corresponding periodic and topological solutions. For instance, it is only when $k = 1$ that we have BPS solutions; for $k \neq 1$ the periodic solutions will never solve the first-order equations. See refs. [17, 18] for more detais on this issue in the case of a single scalar field.

3.3 Energy and stability

In this subsection we introduce another reasoning, that leads to another condition to be fulfilled by the classical solutions in order to make them stable against decaying into less energetic solutions. This reasoning is simple and can be interesting, mainly when one is unable to implement the standard investigation of stability analytically. The reasoning is based on the presence of stable BPS solutions, which solve the first-order equations, and so it does not work for periodic solutions.

Let us consider the parity preserving model defined with H_1. In this case the non-BPS solution obtained with $\phi=0$ has energy $E=(4/3)\mu r^{3/2}a^3$, and connects the two vacuum states $(0,a\sqrt{r})$ and $(0,-a\sqrt{r})$. However, these two vacuum states can be also connected by considering BPS solutions that make use of the vacuum $(a,0)$ or $(-a,0)$. Evidently, there are two degenerate possibilities of connecting the vacua $(0,a\sqrt{r})$ and $(0,-a\sqrt{r})$: one uses $(-a,0)$ and the other $(a,0)$, choosing the left and right path, respectively. The energy of these two BPS solutions can be calculated easily — see sec. 1. The result is $(4/3)\mu r a^3$. We compare this with the energy $E=(4/3)\mu r^{3/2}a^3$ of the non-BPS solution to see that it is only for $r\leq 1$ that the non-BPS defect does not decay into a pair of BPS defects. Interestingly, this is the same result we have already obtained using the standard investigation of classical or linear stability.

4. High temperature effects

Owing to the possibility of applications to cosmology, let us now study the effective potential at finite temperature for the models introduced in this work. The effective potential allows introducing the conditions for symmetry restoration, to identify the symmetric phase, the phase in which the system supports no topological defects anymore. Here the investigations are done in the (3+1) dimensional space-time, and deal with constant and uniform field configurations. The one-loop effective potential and the corresponding finite temperature effects can be calculated according to the standard investigations [26]. It can be written as, keeping only the high temperature contributions [8],

$$U^T(\phi,\chi)=U(\phi,\chi)+\frac{T^2}{24}(U_{\phi\phi}+U_{\chi\chi})\,. \tag{4.1}$$

The above expression for the effective potential shows that the thermal corrections add to mass terms in the following form

$$\begin{aligned} m^2_{\phi\phi}(T) &= U_{\phi\phi}(0,0)+\frac{T^2}{24}(U_{\phi\phi\phi\phi}+U_{\chi\chi\phi\phi})\,,\\ m^2_{\chi\chi}(T) &= U_{\chi\chi}(0,0)+\frac{T^2}{24}(U_{\phi\phi\chi\chi}+U_{\chi\chi\chi\chi})\,,\\ m^2_{\phi\chi}(T) &= U_{\phi\chi}(0,0)+\frac{T^2}{24}(U_{\phi\phi\phi\chi}+U_{\chi\chi\chi\phi})\,. \end{aligned} \tag{4.2}$$

It is convenient to investigate the general model, defined with the potential given by eq. (2.4). The effective potential at finite temperature in this case can be written as

$$\begin{aligned} U^T(\phi,\chi) = {}& \frac{1}{2}(\lambda^2+\nu^2)\phi^4+2\nu(\lambda+\mu)\phi^3\chi+(\lambda\mu+2\nu^2+\sigma\nu+2\mu^2)\phi^2\chi^2+\\ &+2\mu(\nu+\sigma)\phi\chi^3+\frac{1}{2}(\mu^2+\sigma^2)\chi^4+\frac{1}{2}m^2_{\phi\phi}(T)\phi^2+\frac{1}{2}m^2_{\chi\chi}(T)\chi^2+\\ &+m^2_{\phi\chi}(T)\phi\chi+\frac{T^2}{24}\left[m^2_{\phi\phi}(0)+m^2_{\chi\chi}(0)\right]+\frac{1}{2}(\lambda^2a^4+\sigma^2b^4)\,, \end{aligned} \tag{4.3}$$

where the mass parameters at finite and zero temperature are given by

$$m^2_{\phi\phi}(T) = m^2_{\phi\phi}(0) + \frac{T^2}{6}(3\lambda^2 + 5\nu^2 + \lambda\mu + \sigma\nu + 2\mu^2)\,, \tag{4.4}$$

$$m^2_{\chi\chi}(T) = m^2_{\chi\chi}(0) + \frac{T^2}{6}(3\sigma^2 + \lambda\mu + 2\nu^2 + \sigma\nu + 5\mu^2)\,, \tag{4.5}$$

$$m^2_{\phi\chi}(T) = m^2_{\phi\chi}(0) + \frac{T^2}{2}(\lambda\nu + \sigma\mu + 2\mu\nu)\,, \tag{4.6}$$

$$m^2_{\phi\phi}(0) = -2(\lambda^2 a^2 + \sigma b^2\nu)\,, \tag{4.7}$$

$$m^2_{\chi\chi}(0) = -2(\lambda a^2\mu + \sigma^2 b^2)\,, \tag{4.8}$$

$$m^2_{\phi\chi}(0) = -2(\lambda a^2\nu + \sigma b^2\mu)\,. \tag{4.9}$$

In order to ensure that the potential mantains the $Z_2 \times Z_2$ parity symmetry at finite temperature we impose, besides the restrictions already found at zero temperature,

$$\lambda\nu + \sigma\mu + 2\mu\nu = 0\,. \tag{4.10}$$

We notice that this condition is fully satisfied by the choices $\nu = \sigma = 0$ or $\lambda = \mu = 0$ corresponding to H_1 or H_2, respectively. This result indicates that the existing solutions of second-order equations at zero temperature may persist when the system is in equilibrium with a thermal bath. However, we have been unable to write the finite temperature effective potential $U^T(\phi,\chi)$ via some generalized function $H^T(\phi,\chi)$ in the form (1.4) that appears at zero temperature. Although we have no explicit proof, we argue that the thermal effects destroy the Bogomol'nyi bound, together with the possibility of finding BPS defects at finite temperature.

To define the masses properly and to introduce the critical temperatures we rotate the plane (ϕ,χ) to the plane (ϕ_+,ϕ_-) that diagonalizes the mass matrix. In this case we have $m^2_+(T) > 0$ and $m^2_-(T) > 0$ as the conditions for symmetry restoration in each one of the two independet field directions ϕ_+ and ϕ_-, respectively. The critical temperatures are obtained in the limit where these masses vanish, and here they are given by

$$(T^c_\pm)^2 = \frac{12}{\Lambda_1\Lambda_2 - \Lambda_3^2}\left[-(m_1\Lambda_2 + m_2\Lambda_1 - 2m_3\Lambda_3) \pm \sqrt{\Delta}\right], \tag{4.11}$$

where we have set

$$\Delta = (m_1\Lambda_2 - m_2\Lambda_1)^2 + 4m_1m_2\Lambda_3^2 - 4m_3(m_1\Lambda_2\Lambda_3 + m_2\Lambda_1\Lambda_3 - m_3\Lambda_1\Lambda_2)\,. \tag{4.12}$$

The parameters m_i and Λ_i are derivatives of $U(\phi,\chi)$ evaluated at the point $(0,0)$, and are given by

$$\begin{aligned} m_1 &= U_{\phi\phi}\,, & m_2 &= U_{\chi\chi}\,, & m_3 &= U_{\phi\chi}\,,\\ \Lambda_1 &= U_{\phi\phi\phi\phi} + U_{\phi\phi\chi\chi}, & \Lambda_2 &= U_{\chi\chi\chi\chi} + U_{\phi\phi\chi\chi}, & \Lambda_3 &= U_{\phi\phi\phi\chi} + U_{\chi\chi\chi\phi}\,. \end{aligned} \tag{4.13}$$

Let us consider for instance a model defined by the general potential $U(\phi, \chi)$ given by eq. (2.4) but with the restrictions $b^2 = ra^2$, $\nu = r\sigma$, and $\lambda = r\mu$, so that its vacuum states are $(\pm a, 0)$ and $(0, \pm a\sqrt{r})$. The critical temperatures in this case are given by

$$
\begin{aligned}
(T^c_\pm)^2 = &\frac{12a^2r(r+1)[(r+2)\sigma^2 - (2r+1)\mu^2]}{(5r+1)(2r^2+r+3)\sigma^2 - (r+5)(3r^2+r+2)\mu^2} \pm \\
&\pm \frac{12a^2r(r-1)\sqrt{(r-1)^2(\sigma^4+\mu^4) + (6r^2+4r+6)\mu^2\sigma^2}}{(5r+1)(2r^2+r+3)\sigma^2 - (r+5)(3r^2+r+2)\mu^2} \, .
\end{aligned}
\tag{4.14}
$$

Then, to find the critical temperatures for the parity preserving model H_1 we just take $\sigma = 0$ on the above result so that we have

$$
\begin{aligned}
(T^c_+)^2 &= \frac{12r^2a^2}{3r^2+r+2} \, , \\
(T^c_-)^2 &= \frac{12ra^2}{r+5} \, .
\end{aligned}
\tag{4.15}
$$

This result coincide with the one recently found in ref. [8].

It is also interesting to compute the critical temperatures for the model defined by H_2, eq. (2.10). They are also obtained from eq. (4.14), but now imposing $\mu = 0$. Here we get

$$
\begin{aligned}
(T^c_+)^2 &= \frac{12ra^2}{5r+1} \, , \\
(T^c_-)^2 &= \frac{12ra^2}{2r^2+r+3} \, .
\end{aligned}
\tag{4.16}
$$

We notice that these results for H_2 are compatible with the ones obtained for H_1 since they are connected via the identifications: $ra^2 \leftrightarrow a^2$ and $r \leftrightarrow 1/r$.

The existence of two critical temperatures is due to the fact that the systems we are studing have two degrees of freedom and as a consequence their symmetries are restored separately for each one of the two independent field directions.

5. Comments and conclusions

In this paper we have dealt with systems of two coupled real scalar fields, searching for and investigating the corresponding classical or linear stability of the basic topological and periodic solutions when one of the two fields is set to zero. Our investigation is related to the possibility of nesting defects inside defects in 3+1 spacetime dimensions. However, in $3+1$ dimensions renormalization restricts interactions to the fourth power in the polynomial potential, and so we have been mostly concerned with kinks of the hyperbolic tangent type. Such kinks have the property of vanishing at its own core or center, and so at the core of the host defect the system is reduced to a system of a single field, which is still able to generate the other defect, the defect to be nested inside the host defect.

We have shown that the $Z_2 \times Z_2$ parity symmetry is necessary to the formation of defects inside defects of this kind. As we have seen, the two basic defects that appears when one of the two fields vanishes can be summarized as follows:

model	symmetry	topological solutions	nontopological solutions
H_1	$Z_2 \times Z_2$	$(\tanh x, 0)$/BPS $(0, \tanh x)$/non-BPS	(sn x, 0) (0, sn x)
H_2	$Z_2 \times Z_2$	$(\tanh x, 0)$/non-BPS $(0, \tanh x)$/BPS	(sn x, 0) (0, sn x)
H_1 or H_2 $(r = 1)$	Z_4	$(\tanh x, 0)$/BPS $(0, \tanh x)$/BPS	(sn x, 0) (0, sn x)

We recall that in the last case, for $r = 1$ the two $Z_2 \times Z_2$ models colapse to the same Z_4 model, but this model does not describe a system of two *coupled* fields anymore [8].

We have also investigated the presence of periodic solutions when one of the two fields is set to zero. These solutions are direct extensions of the sphalerons found in [17] in the case of a single real scalar field. The basic motivation for doing this is to enlarge the scope of the paper, since now one can mix topological and periodic defects, for instance allowing sphalerons to be nested inside domain walls. Furthermore, since the systems we have considered can be seen as real bosonic portions of supersymmetric models, it seems also interesting to investigate the behavior of fermions in such models.

The high temperature results show the presence of two critical temperatures, signalling symmetry restoration in each one of the two independent field directions. Here the thermal effects seems to destroy the Bogomol'nyi bound, making the BPS defect to vanish at finite temperature. Within the standard cosmological evolution the temperature decreases until breaking the symmetry for the first field, allowing the formation of the host defect and later, after the breakdown of the symmetry corresponding to the second field, the second defect appears nested inside the first defect. As we can see, it is possible to built models in which topological defects can nest nontopological sphalerons, introducing instability inside the planar region on the wall, which may be of interest to applications to cosmology. Other interesting directions consider systems of coupled fields to investigate new aspects of inflation [27, 28, 29, 30] and the possibility of having topological defects ending on topological defects [31, 32]. These and other related issues are presently under consideration.

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References

[1] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other topological defects* Cambridge University Press, Cambridge 1994.

[2] M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58** (1995) 477.

[3] E. Witten, *Nucl. Phys.* **B 249** (1985) 557.

[4] G. Lazarides and Q. Shafi, *Phys. Lett.* **B 159** (1985) 261.

[5] R. MacKenzie, *Nucl. Phys.* **B 303** (1988) 149.

[6] J.R. Morris, *Phys. Rev.* **D 49** (1994) 1105; *Phys. Rev.* **D 51** (1995) 697; *Phys. Rev.* **D 52** (1995) 1096.

[7] D. Bazeia, R.F. Ribeiro and M.M. Santos, *Phys. Rev.* **D 54** (1996) 1852.

[8] F.A. Brito and D. Bazeia, *Phys. Rev.* **D 56** (1997) 7869 [hep-th/9706139].

[9] J.R. Morris, *Int. J. Mod. Phys.* **A 13** (1998) 1115 [hep-ph/9707519].

[10] J.D. Edelstein, M.L. Trobo, F.A. Brito and D. Bazeia, *Phys. Rev.* **D 57** (1998) 7561 [hep-th/9707016].

[11] D. Bazeia, M.J. dos Santos and R.F. Ribeiro, *Phys. Lett.* **A 208** (1995) 84;
D. Bazeia and M.M. Santos, *Phys. Lett.* **A 217** (1996) 28;
D. Bazeia, R.F. Ribeiro and M.M. Santos, *Phys. Rev.* **E 54** (1996) 2943.

[12] J.R. Morris and D. Bazeia, *Phys. Rev.* **D 54** (1996) 5217 [hep-ph/9607396];
D. Bazeia and J.R. Morris, *Domain walls and nontopological solitons in systems of coupled fields,* report (unpublished).

[13] M. Cvetic and H. Soleng, *Phys. Rep.* **282** (1997) 159 [hep-th/9604090].

[14] A. Kovner and M. Shifman, *Phys. Rev.* **D 56** (1997) 2396 [hep-th/9702174];
G. Dvali and M. Shifman, *Phys. Lett.* **B 396** (1997) 64, erratum *ibid.* **407** (1997) 452 [hep-th/9612128];
A. Kovner, M. Shifman and A. Smilga, *Phys. Rev.* **D 56** (1997) 7978 [hep-th/9706089];
B. Chibisov and M. Shifman, *Phys. Rev.* **D 56** (1997) 7990, erratum *ibid.* **58** (1998) 109901 [hep-th/9706141].

[15] M. Shifman, *Phys. Rev.* **D 57** (1998) 1258 [hep-th/9708060];
M.B. Voloshim, *Phys. Rev.* **D 57** (1998) 1266 [hep-th/9708067];
M. Shifman and M.B. Voloshin, *Phys. Rev.* **D 57** (1998) 2590 [hep-th/9709137];
I.I. Kogan, A. Kovner and M. Shifman, *Phys. Rev.* **D 57** (1998) 5195 [hep-th/9712046].

[16] M.K. Prasad and C.M. Sommerfield, *Phys. Rev. Lett.* **35** (1975) 760;
E.B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[17] N. S. Manton and T.M. Samols, *Phys. Lett.* **B 207** (1998) 179.

[18] J.-Q. Liang, H.J.W. Müller-Kristen and D.H. Tchrakian, *Phys. Lett.* **B 282** (1992) 105.

[19] Y. Brihaye, S. Giller, P. Kosinski and J. Kunz, *Phys. Lett.* **B 293** (1992) 383.

[20] M. Abramowitz and I.A. Stegun eds., *Handbook of mathematical functions* National Bereau of Standards, Washington D.C. 1972, chap. 16.

[21] R. Jackiw, *Rev. Mod. Phys.* **49** (1977) 681.

[22] R. Rajaraman, *Solitons and instantons* North-Holland, Amsterdam 1982.

[23] D. Bazeia, J.R.S. Nascimento, R.F. Ribeiro and D. Toledo, J. Phys. A **30** (1997) 8157.

[24] P.M. Morse and H. Feshbach, *Methods of mathematical physics* McGraw-Hill, New York 1953 p. 1650.

[25] F. M. Arscott, *Periodic differential equations* Pergamon, Oxford, 1964.

[26] R. Jackiw, *Phys. Rev.* **D 9** (1974) 1686;
L. Dolan and R. Jackiw, *Phys. Rev.* **D 9** (1974) 3320;
S. Weinberg, *Phys. Rev.* **D 9** (1974) 3357.

[27] L.A. Kofman, A.D. Linde, and A.A. Starobinsky, *Phys. Rev. Lett.* **73** (1994) 3195 [hep-th/9405187].

[28] N.J. Cornish and J.J. Levin, *Phys. Rev.* **D 53** (1996) 3022 [astro-ph/9510010].

[29] D. Boyanovsky, H.J. de Vega, and R. Holman, *Erice lectures on inflationary reheating*, to appear in the Proceedings of the 5th. Erice Chalonge School on Astrofundamental Physics, N. Sánchez and A. Zichichi eds., World Scientific, 1997 [hep-ph/9701304].

[30] L.A. Kofman, A.D. Linde, and A.A. Starobinsky, *Phys. Rev.* **D 56** (1997) 3258 [hep-ph/9704452];
P.B. Greene, L.A. Kofman, A.D. Linde and A.A. Starobinsky, *Phys. Rev.* **D 56** (1997) 6175 [hep-ph/9705347].

[31] S.M. Carroll and M. Trodden, *Phys. Rev.* **D 57** (1998) 5189 [hep-th/9711099].

[32] A. Campos, K. Holland and U.-J. Wiese, *Phys. Rev. Lett.* **81** (1998) 2420 [hep-th/9805086].